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CONTINUUM-WISE EXPANSIVE HOMEOMORPHISMS WITH SHADOWING

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Abstract. In this paper, we show that if a continuum-wise expansive homeomorphism on a compact connected metric space is shadowing then it is topologically stable. This generalizes the main result of [4].

1. Introduction

Expansiveness and shadowing are important in the qualitative theory of dynamical systems, and have lots of applications in topological dynamics, ergodic theory, symbolic dynamics and related areas.

Let (X, d) be a compact connected metric space, and let $f : X \to X$ be a homeomorphism. We say that f is *expansive* if there is $e > 0$ such that for any $x, y \in X$ if $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}$ then $x = y$. Roughly speaking, a system is expansive if two orbits cannot remain close to each other under the action of the system. In light of the rich consequence of expansiveness in the dynamics of a systems, it is natural to consider another notion of expansiveness.

Kato [3] introduced the notion of continuum-wise expansiveness as a generalization of the usual concept of expansiveness. Several interesting properties of continuum-wise expansiveness have been obtained elsewhere [1, 2].

For a closed set $\Lambda \subset X$, we denote $diamA = \sup\{d(x, y) : x, y \in A\}$. We say that a subset $C \subset X$ is a *continuum* if it is compact and connected. A *trivial continuum(or singleton)* is a continuum with only one point. We say that a homeomorphism $f : X \to X$ is *continuum-wise*

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expansive if there exists a constant $\alpha > 0$ such that each non trivial continuum C satisfies $\sup_{n\in\mathbb{Z}} diam f^n(C) > \alpha$. Such a constant α is said to be an *expansive constant for* f. The class of continuum-wise expansive homeomorphisms is much larger than the one of expansive homeomorphisms. In fact, the class of continuum-wise expansive homeomorphisms contains many important homeomorphisms which often appear in chaotic topological dynamics and continuum theory, but which are not expansive homeomorphisms. For example, the shift maps of Knaster's indecomposable chainable continua are continuum-wise expansive homeomorphisms, but they are not expansive homeomorphisms as following:

EXAMPLE 1.1. ([2]) (Knaster's indecomposable chainable continua) *Let I denote the unit interval $[0,1]$. For each $n = 2, 3, ...$, let $f_n : I \to I$ be a map defined*

$$f_n(t) = \begin{cases} nt - s & \text{if s is even} \\ -nt + s + 1 & \text{if s is odd} \end{cases}$$

for $t \in [s/n, s + 1/n]$ and $s = 0, 1, \cdots, n-1$. Then $K(n) = (I, f_n)$ is the Knaster's chainable continuum of order n. The shift map $\tilde{f_n}$ is a continuum-wise expansive homeomorphism but is not an expansive homeomorphism.

For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *δ-pseudo orbit* if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is $y \in X$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

When studying topological dynamics it is more appropriate to require the conjugating map to be a homeomorphism and so to classify up to topological conjugacy. Two homeomorphisms $f : X \to X$ and $g : Y \to Y$ are said to be *conjugate* if there exists a homeomorphism $h : Y \to Y$ such that $hf = gh$. We say that $f : X \to X$ is *topologically stable* if for any $\epsilon > 0$ there is $\delta > 0$ such that if $g : X \to X$ is any homeomorphism with $d(f, g) = \sup_{x\in X} d(f(x), g(x)) < \delta$, then there is a continuous map $h : X \to X$ with $hg = fh$ and $d(h, id) < \epsilon$, where $id : X \to X$ stands for the identity homeomorphism.

Walters [4] proved that any expansive homeomorphism on a compact metric space with shadowing is topologically stable. The following theorem generalizes the results by Walters in [4].

THEOREM 1.2. *Let X be a compact connected metric space, and let $f : X \to X$ be a homeomorphism. If f is continuum-wise expansive and shadowing, then it is topologically stable.*

2. Proof of Theorem 1.2

For $x \in X$ and $\delta > 0$, we define

$$\Gamma_\delta(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \delta \text{ for all n} \in \mathbb{Z}\}.$$

LEMMA 2.1. *If f is continuum-wise expansive, then there is $\delta > 0$ such that for any $x \in X$, $\Gamma_\delta(x)$ contains no non-trivial continua.*

Proof. Consider a continuum-wise expansive constant $\epsilon > 0$ of f and take $\delta = \frac{\epsilon}{2}$. If $A \subset \Gamma_\delta(x)$ is a connected component then $diam(f^n(A)) \leq 2\delta$ for all $n \in \mathbb{Z}$. Since $\epsilon = 2\delta$ is a continuum-wise expansive constant, we conclude that A is a singleton. Hence every continuum contained in $\Gamma_\delta(x)$ is trivial for all $x \in X$. □

A continuum-wise expansive homeomorphism determines the topology of X in the following sense. The following lemma is fundamental in the proof of our main theorem.

LEMMA 2.2. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism and let $e > 0$ be the continuum-wise expansive constant for f. For any $\epsilon > 0$, there is $n \geq 1$ such that $\max_{|i| \leq n} diam f^i(A) > e$ for any continuum A satisfying $diam A \geq \epsilon$.*

Proof. Let $\epsilon > 0$ be given. To derive a contradiction, we may assume that for any $n \geq 1$, there exists a continuum $A_n \subset X$ such that $diam(f^i(A_n)) < e/2$ for $|i| < n$ and $diam A_n \geq \epsilon$.

Choose $x_n, y_n \in A_n$ satisfying

$$diam A_n = \sup\{(x, y) : x, y \in A_n\} = d(x_n, y_n).$$

Then we know

$$d(f^i(x_n), f^i(y_n)) = diam(f^i(A_n))$$

for $|i| < n$. Assume that $A_n \to A$ as $n \to \infty$. Then $x_n \to x$ and $y_n \to y$ as $n \to \infty$, where $d(x, y) = diam A$. If necessary, we take x_{n_i}, y_{n_i} such that

(i) $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_i}\} \subset \{y_n\}$ and
(ii) $x_{n_i} \to x$ and $y_{n_i} \to y$ as $i \to \infty$.

Since $A_n \to A$ as $n \to \infty$, we know $diam A \geq \epsilon$ and $diam(f^i(A)) < e$ for all $i \in \mathbb{Z}$. Let $\Gamma_e(x) = \{y \in X : d(f^i(x), f^i(y)) \leq e \text{ for all } i \in \mathbb{Z}\}$. It is clear that $A \subset \Gamma_e(x)$. Since f is continuum-wise expansive, by Lemma 2.1 the continuum A should be a singleton. This contradicts with $d(x, y) \geq \epsilon$. □

Lemma 2.3. *Let $f : X \to X$ be a continuum-wise expansive homeomorpism with shadowing, and let $e > 0$ be a continuum-wise constant of f. If $\epsilon < e/2$ and δ corresponds to ϵ as in the definition of the shadowing property, then there is a unique $x \in X$ which ϵ-shadows a given δ-pseudo orbit.*

Proof. Let $\{x_n\}$ be a given δ-pseudo-orbit of f and let x and y be two points that ϵ-shadows $\{x_n\}$. Then one has

$$d(f^i(x), f^i(y)) \le d(f^i(x), x_i) + d(x_i, f^i(y)) < 2\epsilon < e$$

for every $i \in \mathbb{Z}$. Then $y \in \Gamma_e(x)$. By the continuum-wise expansiveness of f, we get $x = y$. □

Proof of Theorem 1.2 Let e be an expansive constant of f and let $\epsilon < \frac{e}{3}$. Choose $\delta > 0$ to correspond to ϵ as in definition of shadowing. Let g be a homeomorphism on X with $d(f, g) < \delta$. Then $\{g^n(x) : n \in \mathbb{Z}\}$ be a δ pseudo-orbit for f since $d(f^n(x), g^n(x)) < \delta$. By applying Lemma 2.3, we can take a unique point denoted by $h(x)$ that f-orbit ϵ traces $g^n(x)$. Then we can contstruct a map $h : X \to X$ satisfying

$$d(f^i(h(x)), g^i(x)) < \epsilon \qquad (*)$$

for all $i \in \mathbb{Z}$. In particular, when $i = 0$ then $d(h(x), x) < \epsilon$. It holds for every $x \in X$ and hence $d(h, Id_X) \le \epsilon$.

Now we will prove that $hg(x) = fh(x)$ for every $x \in X$. Let $x \in X$. Then from (*), one have $d(f^i(h(x)), g^{i+1}(x)) < \epsilon$ for every $i \in \mathbb{Z}$, and

$$d(f^i(fh(x)), g^{i+1}(x)) = d(f^{i+1}h(x), g^{i+1}(x)) < \epsilon$$

for every $i \in \mathbb{Z}$. Then by Lemma 2.3, we get $hg(x) = fh(x)$.

Next we will show that h is continuous. Let $\epsilon_1 > 0$. By Lemma 2.2, there exists $n \ge 1$ such that

$$\max_{|i| \le n} diam f^i(A) > e$$

for any continuum A satisfying $diam A \ge \epsilon_1$. Then whenever $d(f^i(x), f^i(y))$ $< e$ one has $d(x, y) < \epsilon_1$. Choose $\delta_1 > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta_1$, one has $d(g^i(x), g^i(y)) < \frac{e}{3}$ for every $i \in \mathbb{Z}$. Then, for every $x, y \in X$ with $d(x, y) < \delta_1$, we get

$$\begin{aligned} d(fh(x), fh(y)) &= d(hg(x), hg(y)) \\ &\le d(hg(x), g(x)) + d(g(x), g(y)) + d(g(y), hg(y)) \\ &< \epsilon + \frac{e}{3} + \epsilon < e. \end{aligned}$$

This means that h is continuous. □

Walters [4] proved that a topologically stable homeomorphism of a compact manifold of dimension ≥ 2 is shadowing. Moreover, very recently, Artigue and Olivera [1] showed that the following implications hold: expansive $\Rightarrow$ countably-expansive $\Leftrightarrow$ measure-expansive $\Rightarrow$ continuum-wise expansive. Consequently we obtain the following corollary.

COROLLARY 2.4. *Let $f : X \to X$ be a homeomorphism of a compact manifold X of dimension ≥ 2. If f is shadowing, then the followings are pairwise equivalent:*

(a) *f is expansive*
(b) *f is measure expansive*
(c) *f is continuum-wise expansive.*

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